The quasiclassical limit of the symmetry constraint of the KP hierarchy and the dispersionless KP hierarchy with self-consistent sources

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Abstract

For the first time we show that the quasiclassical limit of the symmetry constraint of the KP hierarchy leads to the generalized Zakharov reduction of the dispersionless KP (dKP) hierarchy which has been proved to be result of symmetry constraint of the dKP hierarchy recently. By either regarding the constrained dKP hierarchy as its stationary case or taking the dispersionless limit of the KP hierarchy with self-consistent sources directly, we construct a new integrable dispersionless hierarchy, i.e., the dKP hierarchy with self-consistent sources and find its associated conservation equations (or equations of Hamilton-Jacobi type). Some solutions of the dKP equation with self-consistent sources are also obtained by hodograph transformations.

Keywords: quasiclassical limit; constrained KP hierarchy; (dispersionless) KP hierarchy with self-consistent sources; conservation equation (equation of Hamilton-Jacobi type); Zakharov reduction; hodograph transformation

1 Introduction

The dispersionless integrable hierarchies provide us an interesting type of nonlinear integrable models which have important applications from complex analysis to topological field

theory (see [1-12] and the references therein). In [1, 2, 5, 6, 7], a standard procedure of dispersionless limit of integrable dispersionfull hierarchies is proposed. In this procedure, dispersionless hierarchies arise as the quasiclassical limit of the original dispersionfull Lax equations performed by replacing operators by phase space functions, commutators by Poisson brackets and the role of Lax pair equations by conservation equations (or equations of Hamilton-Jacobi type). A $\bar{\partial}$ scheme of dispersionless hierarchies has been proposed by Konopelchenko et al (see [11, 12] and the references therein). Recently, from this point of view, some important reductions of dispersionless hierarchies are shown to be nothing but symmetry constraints [12]. Also several methods for solving dispersionless hierarchies have been formulated such as twistorial method [7, 13] and hodograph transformation [5, 6]. In [5, 6], from the conservation equations of the dispersionless KP equation, Kodama and Gibbons found exact solutions of it and its reductions by hodograph transformations and obtained general hodograph equations for hydrodynamical type equations.

The soliton equations with self-consistent sources (SESCS) are another type of important integrable models in many fields of physics, such as hydrodynamics, state physics, plasma physics, etc (see [14-18]). For example, the KP equation with self-consistent sources (KPESCS) describes the interaction of a long wave with a short-wave packet propagating on the x,y plane at an angle to each other (see [14] and the references therein). In general, the constrained integrable hierarchy may be viewed as stationary system of the corresponding hierarchy with self-consistent sources and the Lax representation for the latter can be obtained naturally from that for the former [16-18]. In this sense, the soliton hierarchies with self-consistent sources may be viewed as integrable generalizations of the original soliton hierarchies.

In contrast with the dispersionfull integrable hierarchies with self-consistent sources, the dispersionless integrable hierarchies with self-consistent sources have not been studied before. In this paper, we first investigate the quasiclassical limit of the symmetry constraint of the KP hierarchy which leads to the generalized Zakharov reduction of the dKP hierarchy. The latter has recently been proved to be symmetry constraint of the dKP hierarchy using the $\bar{\partial}$ method [12]. So, we find the relation that the symmetry constraint for the Lax function of the dispersionless hierarchy can be obtained by the quasiclassical limit of the symmetry constraint for the Lax operator of the corresponding dispersionfull hierarchy. By either regarding the constrained dKP hierarchy as its stationary case or taking the dispersionless limit of the KP hierarchy with self-consistent sources directly, we construct a new

dispersionless hierarchy, i.e., the dKP hierarchy with self-consistent sources. This hierarchy is also integrable since we can find its associated conservation equations (or equations of Hamilton-Jacobi type). So in this sense, the dKP hierarchy with self-consistent sources may be viewed as an integrable generalization of the dKP hierarchy. Compared with the case of a nonlocal term appearing in the t-part of the Lax pair for the KP equation with self-consistent sources [17], the t-part of the conservation equations for the dKP equation with self-consistent sources possesses rational terms with poles. From the obtained conservation equations, we can solve the dKP hierarchy with self-consistent sources by hodograph transformations.

The paper will be organized as follows. In section 2, we remind some definitions and results about the KP hierarchy with self-consistent sources. In section 3, we take the quasiclassical limit to the symmetry constraint of the KP hierarchy to obtain the symmetry constraint of the dKP hierarchy. The dKP hierarchy with self-consistent sources and its associated conservation equations are obtained. Section 4 is devoted to solving the dKP hierarchy with self-consistent sources by hodograph transformations and some solutions for the dKP equation with self-consistent sources are presented.

2 The KP hierarchy with self-consistent sources

We first review some definitions and results about the KP hierarchy with self-consistent sources in the framework of Sato theory [17-20]. Given a pseudo-differential operator (PDO) of the form

$$L = \partial + u_1(t)\partial^{-1} + u_2(t)\partial^{-2} + ..., \tag{2.1}$$

where $\partial = \partial_x$, $\partial\partial^{-1} = \partial^{-1}\partial = 1$, $t = (t_1 = x, t_2, \cdots)$, the KP hierarchy is defined as

$$\partial_{t_n} L = [B_n, L], \tag{2.2}$$

where $B_n = (L^n)_{\geq 0}$ is the differential part of L^n. The Lax euqation (2.2) is equivalent to the existence of the Baker function ψ such that

$$L\psi = \lambda\psi, \tag{2.3a}$$

$$\partial_{t_n}\psi = B_n\psi, \tag{2.3b}$$

and ψ also satisfies

$$\partial_\lambda\psi = M\psi, \tag{2.4}$$

where M is the Orlov operator of the KP hierarchy. The adjoint Baker function ψ^* satisfies

$$L^*\psi^* = \lambda\psi^*, \tag{2.5a}$$

$$\partial_n\psi^* = -B_n^*\psi^*, \tag{2.5b}$$

$$\partial_\lambda\psi^* = -M^*\psi^*. \tag{2.5c}$$

Making a constraint of the PDO L (2.1) as [21]

$$L^n = (L^n)_{\geq 0} + \sum_1^N q_i(t)\partial^{-1}r_i(t), \quad n \in \mathbb{N} \tag{2.6}$$

where $q_i(t)$ and $r_i(t)$ satisfying

$$q_{i,t_m} = B_m q_i, \quad r_{i,t_m} = -B_m^* r_i, \quad B_m = [(L^n)^{\frac{m}{n}}]_{\geq 0}, \quad i = 1, ..., N, \forall m \in \mathbb{N}, \tag{2.7}$$

we will get the n-constrained KP hierarchy as

$$(L^n)_{t_m} = [B_m, L^n], \tag{2.8a}$$

$$q_{i,t_m} = B_m q_i, \tag{2.8b}$$

$$r_{i,t_m} = -B_m^* r_i, \quad i = 1, ..., N. \tag{2.8c}$$

If we add the term $(B_m)_{t_n}$ to the right side of equation (2.8a), the KP hierarchy with self-consistent sources will be obtained as [17]

$$(B_m)_{t_n} - (L^n)_{t_m} + [B_m, L^n] = 0, \tag{2.9a}$$

$$q_{i,t_m} = B_m q_i, \tag{2.9b}$$

$$r_{i,t_m} = -B_m^* r_i, \quad i = 1, ..., N. \tag{2.9c}$$

As many cases in (1+1)-dimension (see [16] and the references therein), the n-constrained KP hierarchy may be considered as the stationary one of the KP hierarchy with self-consistent sources if "t_n" is viewed as the evolution variable and the Lax representation for the KP hierarchy with self-consistent sources can be obtained naturally from that for the constrained KP hierarchy [17]. The most important equation in the hierarchy is the KP equation with self-consistent sources obtained when $n = 3$, $m = 2$ in (2.9), [14]

$$[u_{1,t} - 3u_1u_{1,x} - \frac{1}{4}u_{1,xxx} + \sum_{i=1}^N (q_ir_i)_x]_x - \frac{3}{4}u_{1,yy} = 0, \tag{2.10a}$$

$$q_{i,y} = q_{i,xx} + 2u_1 q_i, \tag{2.10b}$$

$$r_{i,y} = -r_{i,xx} - 2u_1 r_i, \quad i = 1, ..., N \tag{2.10c}$$

where $t = t_3, y = t_2$. With (2.10b) and (2.10c), (2.10a) will be obtained by the compatibility of the following linear equations (Lax pair)[17]

$$\psi_y = \psi_{xx} + 2u_1\psi, \tag{2.11a}$$

$$\psi_t = \psi_{xxx} + 3u_1\psi_x + \frac{3}{2}(u_{1,x} + (\partial^{-1}u_{1,y}))\psi + \sum_{i=1}^{N} q_i\partial^{-1}(r_i\psi). \tag{2.11b}$$

3 Dispersionless limit

Following the standard procedure of dispersionless limit introduced in [1, 2, 5, 6, 7], we will get the dispersionless counterpart of (2.9) which can be regarded as the dispersionless KP hierarchy with self-consistent sources. Simply taking $T_n = \epsilon t_n$ and thinking of $u_n(\frac{T}{\epsilon}) = U_n(T) + O(\epsilon)$ as $\epsilon \to 0$, L in (2.1) changes into

$$L_\epsilon = \epsilon\partial + \sum_{i=1}^{\infty} u_i(\frac{T}{\epsilon})(\epsilon\partial)^{-i} = \epsilon\partial + \sum_{i=1}^{\infty}(U_i(T) + O(\epsilon))(\epsilon\partial)^{-i}, \quad \partial = \partial_X, \quad X = \epsilon x. \tag{3.1}$$

The constraint (2.6) now changes into

$$L_\epsilon^n = B_{\epsilon n} + \sum_{i=1}^{N} q_i(\frac{T}{\epsilon})(\epsilon\partial)^{-1}r_i(\frac{T}{\epsilon}), \quad B_{\epsilon n} = (L_\epsilon^n)_{\geq 0}, \tag{3.2}$$

where $q_i(\frac{T}{\epsilon})$ and $r_i(\frac{T}{\epsilon})$ satisfy

$$\epsilon[q_i(\frac{T}{\epsilon})]_{T_m} = B_{\epsilon m}q_i(\frac{T}{\epsilon}), \quad \epsilon[r_i(\frac{T}{\epsilon})]_{T_m} = -B_{\epsilon m}^* r_i(\frac{T}{\epsilon}), \quad B_{\epsilon m} = [(L_\epsilon^n)^{\frac{m}{n}}]_{\geq 0}, \quad i = 1, ..., N, \tag{3.3}$$

and the counterpart of (2.8) is

$$\epsilon(L_\epsilon^n)_{T_m} = [B_{\epsilon m}, L_\epsilon^n], \tag{3.4a}$$

$$\epsilon[q_i(\frac{T}{\epsilon})]_{T_m} = B_{\epsilon m}q_i(\frac{T}{\epsilon}), \tag{3.4b}$$

$$\epsilon[r_i(\frac{T}{\epsilon})]_{T_m} = -B_{\epsilon m}^* r_i(\frac{T}{\epsilon}), \quad i = 1, ..., N. \tag{3.4c}$$

It is proved in [7] that

$$\mathcal{L} = \sigma^\epsilon(L_\epsilon) = p + \sum_{i=1}^{\infty} U_i(T)p^{-i}. \tag{3.5}$$

is a solution of the dKP hierarchy, i.e., satisfies

$$\partial_{T_n}\mathcal{L} = \{\mathcal{B}_n, \mathcal{L}\}, \tag{3.6}$$

where σ^ϵ denotes the principal symbol [7], the Poisson bracket is defined as

$$\{A(p,x), B(p,x)\} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial p}, \tag{3.7}$$

and $\mathcal{B}_n = (\mathcal{L})^n_{\geq 0}$ now refers to powers of p.
The dKP hierarchy can be also written in the zero-curvature form

$$\frac{\partial \mathcal{B}_n}{\partial T_m} - \frac{\partial \mathcal{B}_m}{\partial T_n} + \{\mathcal{B}_n, \mathcal{B}_m\} = 0. \tag{3.8}$$

In [7], from (2.3) and (2.4) (with L, B_n, M and ∂_n replaced by L_ϵ, $B_{\epsilon n}$, M_ϵ and $\epsilon\partial_{T_n}$ respectively), it has been proved that $\psi(\frac{T}{\epsilon})$ has the following WKB asymptotic expansion as $\epsilon \to 0$

$$\psi(\frac{T}{\epsilon}) = exp[\frac{1}{\epsilon}S(T,\lambda) + O(1)], \quad \epsilon \to 0. \tag{3.9}$$

In order to take the quasiclassical limit of the constraint (3.2), we also need to find the asymptotic form of the adjoint Baker function. Similarly like the proof in [7], from (2.5), we can find $\psi^*(\frac{T}{\epsilon})$ has the following WKB asymptotic expansion

$$\psi^*(\frac{T}{\epsilon}) = exp[-\frac{1}{\epsilon}S(T,\lambda) + O(1)], \quad \epsilon \to 0. \tag{3.10}$$

From (2.3b) and (3.9), we obtain a hierarchy of conservation equations for the momentum function $p = \frac{\partial S}{\partial X}$,

$$\frac{\partial p}{\partial T_n} = \frac{\partial \mathcal{B}_n(p)}{\partial X}, \tag{3.11}$$

the compatibility of which, i.e., $\frac{\partial^2 p}{\partial T_n \partial T_m} = \frac{\partial^2 p}{\partial T_m \partial T_n}$ implies the dKP hierarchy (3.8).
Regarding

$$q_i(\frac{T}{\epsilon}) = \psi(\frac{T}{\epsilon}, \lambda = \lambda_i) \sim exp[\frac{S(T,\lambda_i)}{\epsilon} + \alpha_{i1} + O(\epsilon)], \quad \epsilon \to 0, \tag{3.12a}$$

$$r_i(\frac{T}{\epsilon}) = \psi^*(\frac{T}{\epsilon}, \lambda = \lambda_i) \sim exp[-\frac{S(T,\lambda_i)}{\epsilon} + \alpha_{i2} + O(\epsilon)], \quad \epsilon \to 0, \quad i = 1, ..., N, \tag{3.12b}$$

we will find that

$$\begin{aligned} & q_i(\tfrac{T}{\epsilon})(\epsilon\partial)^{-1} r_i(\tfrac{T}{\epsilon}) \\ = \; & e^{\alpha_{i1}+\alpha_{i2}}[(\epsilon\partial)^{-1} + (\tfrac{\partial S(T,\lambda_i)}{\partial X} + O(\epsilon))(\epsilon\partial)^{-2} + ((\tfrac{\partial S(T,\lambda_i)}{\partial X})^2 + O(\epsilon))(\epsilon\partial)^{-3} + \cdots], \quad \epsilon \to 0. \end{aligned}$$

Taking the principal symbol of both sides of (3.2), we have

$$\begin{aligned}
\mathcal{L}^n &= \mathcal{B}_n + \sum_{i=1}^{N} e^{\alpha_{i1}+\alpha_{i2}}[p^{-1} + S_X(T,\lambda_i)p^{-2} + S_X^2(T,\lambda_i)p^{-3} + \cdots] \\
&= \mathcal{B}_n + \sum_{i=1}^{N} \frac{v_i}{p-p_i} \qquad (3.13)
\end{aligned}$$

where $\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}$ and

$$v_i = e^{\alpha_{i1}+\alpha_{i2}}, \quad p_i = S_X(T,\lambda_i). \qquad (3.14)$$

The constraint (3.13) is well known and is often called Zakharov reduction when $n = 1$ [2, 9, 10, 12]. In [12], using the $\bar{\partial}$ method, the authors demonstrated that (3.13) is a result of symmetry constraint of the dKP hierarchy. Here we have shown that (3.13) can be obtained by the dispersionless limit of (2.6), i.e., the dispersionless limit of symmetry constraint for the Lax operator of the dispersionfull hierarchy leads to the symmetry constraint for the Lax function of the dispersionless hierarchy. From (3.3), (3.12), (3.14) and by a tedious computation, we obtain the following equations of hydrodynamical type

$$p_{i,T_k} = [\mathcal{B}_k(p)|_{p=p_i}]_X, \qquad (3.15a)$$

$$v_{i,T_k} = [v_i(\frac{\partial}{\partial p}\mathcal{B}_k(p))|_{p=p_i}]_X, \quad i = 1, \cdots, N. \qquad (3.15b)$$

Taking the dispersionless limit of (3.4), we will get the constrained dKP hierarchy reduced by (3.13)

$$(\mathcal{L}^n)_{T_k} = \{\mathcal{B}_k, \mathcal{L}^n\}, \qquad (3.16a)$$

$$p_{i,T_k} = [\mathcal{B}_k(p)|_{p=p_i}]_X, \qquad (3.16b)$$

$$v_{i,T_k} = [v_i(\frac{\partial}{\partial p}\mathcal{B}_k(p))|_{p=p_i}]_X, \quad i = 1, ..., N. \qquad (3.16c)$$

Adding the term $(\mathcal{B}_k)_{T_n}$ to the right hand side of (3.16a), or taking the dispersionless limit of (2.9) directly, we will obtain the dKP hierarchy with self-consistent sources as

$$(\mathcal{B}_k)_{T_n} - (\mathcal{L}^n)_{T_k} + \{\mathcal{B}_k, \mathcal{L}^n\} = 0, \qquad (3.17a)$$

$$p_{i,T_k} = [\mathcal{B}_k(p)|_{p=p_i}]_X, \qquad (3.17b)$$

$$v_{i,T_k} = [v_i(\frac{\partial}{\partial p}\mathcal{B}_k(p))|_{p=p_i}]_X, \quad i = 1, ..., N. \qquad (3.17c)$$

If "T_n" is viewed as the evolution variable, (3.16) may be regarded as the stationary system of (3.17) like the dispersionfull case. It is not difficult to prove that under (3.17b) and (3.17c),

the equation (3.17a) will be obtained by the compatibility of the following conservation equations

$$p_{T_k} = (\mathcal{B}_k(p))_X, \tag{3.18a}$$

$$p_{T_n} = (\mathcal{L}^n(p))_X = [\mathcal{B}_n(p) + \sum_{i=1}^{N} \frac{v_i}{p - p_i}]_X. \tag{3.18b}$$

For example, when $n = 3$, $k = 2$,

$$\mathcal{L}^3 = p^3 + 3U_1 p + 3U_2 + \sum_{i=1}^{N} \frac{v_i}{p - p_i}, \quad \mathcal{B}_2 = p^2 + 2U_1,$$

(3.17) becomes the dKP equation with self-consistent sources (U_2 is eliminated by $U_{2,X} = \frac{1}{2}U_{1,Y}$ and $T = T_3$, $Y = T_2$)

$$(U_{1,T} - 3U_1U_{1,X} + \sum_{i=1}^{N} v_{i,X})_X = \frac{3}{4}U_{1,YY}, \tag{3.19a}$$

$$p_{i,Y} = (p_i^2 + 2U_1)_X, \tag{3.19b}$$

$$v_{i,Y} = 2(v_i p_i)_X, \quad i = 1, ..., N. \tag{3.19c}$$

Under (3.19b) and (3.19c), the compatibility of the following conservation equations give rise to (3.19a)

$$p_Y = (p^2 + 2U_1)_X = 2pp_X + 2U_{1,X}, \tag{3.20a}$$

$$p_T = (p^3+3U_1p+3U_2+\sum_{i=1}^{N} \frac{v_i}{p - p_i})_X = 3p^2p_X+3(U_1p)_X+3U_{2,X}+\sum_{i=1}^{N} \frac{v_{i,X}}{p - p_i}-\sum_{i=1}^{N} \frac{v_i(p_X - p_{i,X})}{(p - p_i)^2}. \tag{3.20b}$$

where $U_{2,X} = \frac{1}{2}U_{1,Y}$.

4 Hodograph solutions

Motivated by the dKP case, we would like to solve (3.17) by hodograph transformation provided the conservation equations (3.18). Following [5], one can consider the M-reductions of (3.18) so that the momentum function p and v_i, p_i,$i = 1, ..., N$ depend only on a set of functions W =(W_1,...,W_M) with $W_1 = U_1$ and (W_1,...,W_M) satisfy commuting flows

$$\frac{\partial W}{\partial T_n} = A_n(W)\frac{\partial W}{\partial X}, \quad n \geq 2 \tag{4.1}$$

where the $M \times M$ matrices A_n are functions of $(W_1, ..., W_M)$ only. In the following, we shall take the dKP equation with self-consistent sources (3.19) for example and show its solutions in the cases of $M = 1$ and $M = 2$.

1. $M = 1$

In this case, $p = p(U_1)$, $v_i = v_i(U_1)$, $p_i = p_i(U_1)$ and (4.1) becomes

$$U_{1,Y} = A(U_1)U_{1,X}, \quad U_{1,T} = B(U_1)U_{1,X}. \tag{4.2}$$

From (3.19b), (3.19c) and (3.20), we will get the following relations respectively

$$\frac{dp_i}{dU_1}(\frac{A(U_1)}{2} - p_i) = 1, \tag{4.3a}$$

$$\frac{dv_i}{dU_1}(\frac{A(U_1)}{2} - p_i) = v_i\frac{dp_i}{dU_1}, \tag{4.3b}$$

$$\frac{dp}{dU_1}(\frac{A(U_1)}{2} - p) = 1, \tag{4.3c}$$

$$\frac{dp}{dU_1}B(U_1) = 3p^2\frac{dp}{dU_1} + 3p + 3U_1\frac{dp}{dU_1} + 3\frac{dU_2}{dU_1} + \sum_{i=1}^{N}\frac{\frac{dv_i}{dU_1}}{p - p_i} - \sum_{i=1}^{N} v_i\frac{(\frac{dp}{dU_1} - \frac{dp_i}{dU_1})}{(p - p_i)^2}, \tag{4.3d}$$

which implies

$$B = 3U_1 + \frac{3}{4}A^2 - \sum_{i=1}^{N}\frac{dv_i}{dU_1}, \quad A = 2\frac{dU_2}{dU_1}. \tag{4.4}$$

It is easy to verify that with (4.3) and (4.4), (4.2) are compatible. Making the hodograph transformations with the change of variables $(X, Y, T) \to (U_1, Y, T)$ and $X = X(U_1, Y, T)$, we will get the following hodograph equations for X,

$$\frac{\partial X}{\partial Y} = -A, \quad \frac{\partial X}{\partial T} = -B = -3U_1 - \frac{3}{4}A^2 + \sum_{i=1}^{N}\frac{dv_i}{dU_1}, \tag{4.5}$$

which can be easily integrated as follows

$$X + A(U_1)Y + (3U_1 + \frac{3}{4}A(U_1)^2 - \sum_{i=1}^{N}\frac{dv_i}{dU_1})T = F(U_1), \tag{4.6}$$

where $F(U_1)$ is an arbitrary function of U_1.

If we choose $A(U_1) = C_0 = \text{const}$, $F(U_1) = \beta U_1$, from (4.6), (4.3a) and (4.3b), we get an implicit solution as

$$X + C_0Y + [3U_1 + \frac{3}{4}C_0^2 - \sum_{i=1}^{N} c_i(\frac{C_0^2}{4} - 2(U_1 + d_i))^{-\frac{3}{2}}]T = \beta U_1, \tag{4.7a}$$

$$v_i = c_i[\frac{C_0^2}{4} - 2(U_1 + d_i)]^{-\frac{1}{2}}, \tag{4.7b}$$

$$p_i = \frac{C_0}{2} \pm \sqrt{\frac{C_0^2}{4} - 2(U_1 + d_i)}, \quad i = 1, ..., N, \tag{4.7c}$$

where c_i,d_i,$i = 1, ..., N$ are constants. If $d_1 = d_2 = \cdots = d_N$ and $\sum_{i=1}^N c_i = 0$, (4.7a) degenerates to the solution of dKP equation [5].

If we assume $v_1 = v_2 = \cdots = v_N$ and $\sum_{i=1}^N \frac{dv_i}{dU_1} = 3U_1$, i.e., $\frac{dv_i}{dU_1} = \frac{3}{N}U_1$, and taking $F(U_1) = 0$, by a direct computation, we will obtain an explicit solution of the dKP equation with self-consistent sources (3.19) as follows

$$U_1 = \frac{4Y^2 - 6TX \pm 4Y\sqrt{Y^2 - 3TX}}{225T^2},$$

$$v_i = \frac{3}{2N}U_1^2,$$

$$p_i = 2\sqrt{2U_1}, \quad i = 1, ..., N.$$

2. $M = 2$

In this case we denote $W_1 = U_1$, $W_2 = W$, then $v_i = V_i(U_1, W)$, $p_i = p_i(U_1, W)$ and $p = p(U_1, W)$ with the commuting flows

$$\begin{pmatrix} U_1 \\ W \end{pmatrix}_Y = A \begin{pmatrix} U_1 \\ W \end{pmatrix}_X, \qquad \begin{pmatrix} U_1 \\ W \end{pmatrix}_T = B \begin{pmatrix} U_1 \\ W \end{pmatrix}_X, \tag{4.8}$$

where $A = (A_{ij})$ and $B = B_{ij}$ are 2×2 matrix functions of U_1 and W. Requiring $U_{1,x}$ and W_x are independent, (3.19b), (3.19c) and (3.20) give rise to the following relations respectively

$$(\frac{\partial p_i}{\partial U_1}, \frac{\partial p_i}{\partial W})A = (2 + 2p_i\frac{\partial p_i}{\partial U_1}, 2p_i\frac{\partial p_i}{\partial W}), \tag{4.9a}$$

$$(\frac{\partial v_i}{\partial U_1}, \frac{\partial v_i}{\partial W})A = (2\frac{\partial(v_i p_i)}{\partial U_1}, 2\frac{\partial(v_i p_i)}{\partial W}), \tag{4.9b}$$

$$(\frac{\partial p}{\partial U_1}, \frac{\partial p}{\partial W})A = 2p(\frac{\partial p}{\partial U_1}, \frac{\partial p}{\partial W}) + (2, 0), \tag{4.9c}$$

$$\begin{aligned} (\tfrac{\partial p}{\partial U_1}, \tfrac{\partial p}{\partial W})B \ = \ & 3p^2(\tfrac{\partial p}{\partial U_1}, \tfrac{\partial p}{\partial W}) + 3(\tfrac{\partial(U_1 p)}{\partial U_1}, \tfrac{\partial(U_1 p)}{\partial W}) + 3(\tfrac{\partial U_2}{\partial U_1}, \tfrac{\partial U_2}{\partial W}) \\ & + \textstyle\sum_{i=1}^N \tfrac{1}{p-p_i}(\tfrac{\partial v_i}{\partial U_1}, \tfrac{\partial v_i}{\partial W}) - \textstyle\sum_{i=1}^N \tfrac{v_i}{(p-p_i)^2}(\tfrac{\partial p}{\partial U_1} - \tfrac{\partial p_i}{\partial U_1}, \tfrac{\partial p}{\partial W} - \tfrac{\partial p_i}{\partial W}), \end{aligned} \tag{4.9d}$$

which implies $A(U_1, W)$ and $B(U_1, W)$ must satisfy

$$B = \frac{3}{4}A^2 + 3U_1 I - \sum_{i=1}^N \frac{\partial v_i}{\partial U_1} I - \begin{pmatrix} 0 & \sum_{i=1}^N \frac{\partial v_i}{\partial W} \\ \frac{A_{11}}{A_{12}} \sum_{i=1}^N \frac{\partial v_i}{\partial W} & \frac{A_{22}-A_{11}}{A_{12}} \sum_{i=1}^N \frac{\partial v_i}{\partial W} \end{pmatrix} \tag{4.10}$$

with $A_{11} = 2\frac{\partial U_2}{\partial U_1}$ and $A_{12} = 2\frac{\partial U_2}{\partial W}$.
For simplicity we assume $\frac{\partial v_i}{\partial W} = 0$, $i = 1, ..., N$. Then

$$B = \frac{3}{4}A^2 + 3U_1 I - \sum_{i=1}^{N} \frac{\partial v_i}{\partial U_1} I = \frac{3}{4}(trA)A + 3(U_1 - \frac{1}{4}detA - \frac{1}{3}\sum_{i=1}^{N}\frac{\partial v_i}{\partial U_1})I, \tag{4.11}$$

where the formula $A^2 = (trA)A - (detA)I$ is used.
With (4.11), the compatibility for (4.8) requires A to satisfy

$$\begin{pmatrix} \frac{1}{4}\frac{\partial detA}{\partial W} \\ \frac{\partial(U_1 - \frac{1}{4}detA - \frac{1}{3}\sum_{i=1}^{N}\frac{\partial v_i}{\partial U_1})}{\partial U_1} \end{pmatrix} = A \begin{pmatrix} \frac{1}{4}\frac{\partial trA}{\partial W} \\ -\frac{1}{4}\frac{\partial trA}{\partial U_1} \end{pmatrix}. \tag{4.12}$$

Using the hodograph transformation changing the independent variables (X, Y, T) to (U_1, W, T) with $X = X(U_1, W, T)$ and $Y = Y(U_1, W, T)$, we get

$$\begin{pmatrix} -X_W \\ X_{U_1} \end{pmatrix} = A \begin{pmatrix} Y_W \\ -Y_{U_1} \end{pmatrix}, \qquad \begin{pmatrix} \frac{\partial(X,Y)}{\partial(W,T)} \\ -\frac{\partial(X,Y)}{\partial(U_1,T)} \end{pmatrix} = B \begin{pmatrix} Y_W \\ -Y_{U_1} \end{pmatrix}, \tag{4.13}$$

where $\frac{\partial(X,Y)}{\partial(W,T)} = X_W Y_T - X_T Y_W$. It is not difficult to see that (4.13) has solutions in the form

$$X + 3(U_1 - \frac{1}{4}detA - \frac{1}{3}\sum_{i=1}^{N}\frac{\partial v_i}{\partial U_1})T = F(U_1, W), \tag{4.14a}$$

$$Y + \frac{3}{4}(trA)T = G(U_1, W), \tag{4.14b}$$

where we have required that Y_{U_1} and Y_W are independent, while F and G satisfy the linear equations

$$\begin{pmatrix} -F_W \\ F_{U_1} \end{pmatrix} = A \begin{pmatrix} G_W \\ -G_{U_1} \end{pmatrix}. \tag{4.15}$$

An example of solution is given by

$$A = \begin{pmatrix} W & U_1 \\ 4 & W \end{pmatrix}, \tag{4.16}$$

and

$$v_i = c_i U_1, \quad p_i = \frac{W}{2}, \quad i = 1, \cdots, N, \tag{4.17}$$

with c_i,$i = 1, \cdots, N$ are constants.
(4.14) now becomes

$$X + 3[U_1 - \frac{1}{4}(W^2 - 4U_1) - \frac{1}{3}\sum_{i=1}^{N} c_i]T = F(U_1, W), \tag{4.18a}$$

$$Y + \frac{3}{4} 2WT = G(U_1, W). \tag{4.18b}$$

From (4.15) and $F_{WU_1} = F_{U_1W}$, G must satisfy

$$2G_{U_1} + U_1 G_{U_1U_1} - 4G_{WW} = 0.$$

Taking $G = -W$ and from (4.15), $F = \frac{1}{2}W^2 - 4U_1$ and we obtain a solution of (3.19) as follows

$$U_1 = \frac{Y^2}{2(3T+2)^2} + \frac{(\sum_{i=1}^N c_i)T - X}{2(3T+2)}, \tag{4.19a}$$

$$v_i = c_i U_1 = c_i[\frac{Y^2}{2(3T+2)^2} + \frac{(\sum_{i=1}^N c_i)T - X}{2(3T+2)}], \tag{4.19b}$$

$$p_i = \frac{W}{2} = -\frac{Y}{3T+2}. \quad i = 1, ..., N. \tag{4.19c}$$

(4.19) is a global solution for $T > -\frac{2}{3}$ and (4.19a) degenerates to the solution of the dKP equation [5] when $\sum_{i=1}^N c_i = 0$.

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